Appendix C

Evaluation of the Loss Function

The loss function $L(Q)$ is the expected amount a random variable exceeds a fixed value. For example, if the random variable is demand, then $L(Q)$ is the expected amount demand is greater than Q. See Appendix A, Statistics Tutorial, for a more extensive description of the loss function.

This appendix describes how the loss function of a discrete distribution function can be efficiently evaluated. (Appendix A gives one solution method, but it is inefficient.) If you need to evaluate the loss function of a continuous distribution, then convert the continuous distribution into a discrete distribution by "chopping it up" into many pieces. For example, the standard normal table is the discrete (i.e., "chopped up") version of the continuous standard normal distribution function.

Let N be the number of quantities in the distribution function and let $Q_1, Q_2, Q_3, \ldots, Q_N$ be those quantities. For example, take the empirical distribution function in Chapter 11, repeated here for convenience:

Q	F(Q)	Q	F(Q)	Q	F(Q)
800	0.0303	2,592	0.3636	3,936	0.6970
1,184	0.0606	2,624	0.3939	4,000	0.7273
1,792	0.0909	2,752	0.4242	4,064	0.7576
1,792	0.1212	3,040	0.4545	4,160	0.7879
1,824	0.1515	3,104	0.4848	4,352	0.8182
1,888	0.1818	3,136	0.5152	4,544	0.8485
2,048	0.2121	3,264	0.5455	4,672	0.8788
2,144	0.2424	3,456	0.5758	4,800	0.9091
2,208	0.2727	3,680	0.6061	4,928	0.9394
2,304	0.3030	3,744	0.6364	4,992	0.9697
2,560	0.3333	3,808	0.6667	5,120	1.0000

$F(Q)$ = Probability demand is less than or equal to the quantity Q

With this distribution function, there are 33 quantities, so $N = 33$ and $Q_1 = 800$, $Q_2 = 1{,}184, \ldots,$ and $Q_{33} = 5{,}120$. Furthermore, recall that we use μ to represent expected demand, which in this case is $\mu = 3{,}192$.

We can recursively evaluate the loss function, which means we start with $L(Q_1)$ and then use $L(Q_1)$ to evaluate $L(Q_2)$, and then use $L(Q_2)$ to evaluate $L(Q_3)$, and so forth.

The expected lost sales if we order Q_1 (which in this case is 800 units) are

$$L(Q_1) = \mu - Q_1 = 3{,}192 - 800 = 2{,}392$$

Expected lost sales if we order Q_2 are

$$\begin{aligned} L(Q_2) &= L(Q_1) - (Q_2 - Q_1) \times (1 - F(Q_1)) \\ &= 2{,}392 - (1{,}184 - 800) \times (1 - 0.0303) \\ &= 2{,}020 \end{aligned}$$

Expected lost sales if we order Q_3 are

$$\begin{aligned} L(Q_3) &= L(Q_2) - (Q_3 - Q_2) \times (1 - F(Q_2)) \\ &= 2{,}020 - (1{,}792 - 1{,}184) \times (1 - 0.0606) \\ &= 1{,}448 \end{aligned}$$

In general, the ith expected lost sales are

$$L(Q_i) = L(Q_{i-1}) - (Q_i - Q_{i-1}) \times (1 - F(Q_{i-1}))$$

So you start with $L(Q_1) = \mu - Q_1$ and then you evaluate $L(Q_2)$, and then $L(Q_3)$, up to $L(Q_N)$. The resulting table is

Q	F(Q)	L(Q)	Q	F(Q)	L(Q)	Q	F(Q)	L(Q)
800	0.0303	2,392	2,592	0.3636	841	3,936	0.6970	191
1,184	0.0606	2,020	2,624	0.3939	821	4,000	0.7273	171
1,792	0.0909	1,448	2,752	0.4242	744	4,064	0.7576	154
1,792	0.1212	1,448	3,040	0.4545	578	4,160	0.7879	131
1,824	0.1515	1,420	3,104	0.4848	543	4,352	0.8182	90
1,888	0.1818	1,366	3,136	0.5152	526	4,544	0.8485	55
2,048	0.2121	1,235	3,264	0.5455	464	4,672	0.8788	36
2,144	0.2424	1,160	3,456	0.5758	377	4,800	0.9091	20
2,208	0.2727	1,111	3,680	0.6061	282	4,928	0.9394	8
2,304	0.3030	1,041	3,744	0.6364	257	4,992	0.9697	5
2,560	0.3333	863	3,808	0.6667	233	5,120	1.0000	1

Q = Order quantity
$F(Q)$ = Probability demand is less than or equal to the order quantity
$L(Q)$ = Loss function (the expected amount demand exceeds Q)

With this empirical distribution example, the quantities differ by more than one unit, for example, $Q_2 - Q_1 = 384$. Now suppose the demand forecast is the Poisson distribution with mean 1.25. The distribution function is given in Table A.1 but is repeated here for convenience:

Q	f(Q)	F(Q)
0	0.28650	0.28650
1	0.35813	0.64464
2	0.22383	0.86847
3	0.09326	0.96173
4	0.02914	0.99088
5	0.00729	0.99816
6	0.00152	0.99968
7	0.00027	0.99995
8	0.00004	0.99999
9	0.00001	1.00000

Now we have $Q_1 = 0$, $Q_2 = 1$, and so forth. We find the expected lost sales with the same process: $L(Q_1) = 1.25 - 0 = 1.25$ and

$$\begin{aligned} L(Q_2) &= L(Q_1) - (Q_2 - Q_1) \times (1 - F(Q_1)) \\ &= 0.53650 - (2 - 1) \times (1 - 0.64469) \\ &= 0.18114 \end{aligned}$$

Completing the table yields

Q	f(Q)	F(Q)	L(Q)
0	0.28650	0.28650	1.25000
1	0.35813	0.64464	0.53650
2	0.22383	0.86847	0.18114
3	0.09326	0.96173	0.04961
4	0.02914	0.99088	0.01134
5	0.00729	0.99816	0.00221
6	0.00152	0.99968	0.00038
7	0.00027	0.99995	0.00006
8	0.00004	0.99999	0.00001
9	0.00001	1.00000	0.00000